

Which Kind of Two-Particle States Can Be Teleported through a Three-Particle Quantum Channel? *

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Abstract

The use of a three-particle quantum channel to teleport entangled states through a slight modification of the standard teleportation procedure is studied. It is shown that it is not possible to perform successful teleportation of an arbitrary and unknown two-particle entangled state, following our version of the standard teleportation procedure. On the contrary, it is shown which, and in how many different ways, particular classes of two-particle states can be teleported.

Key words: Teleportation, Bell measurement, Entanglement.

1 Introduction.

The quantum teleportation process permits one to transmit unknown quantum states from a sender to a receiver which are spatially separated.

The classical idea of teleportation, as portrayed in science fiction novels and movies, involves a complete dematerialization of an object positioned in place A and its reappearance at a distant place B.

By contrast, quantum teleportation differs from this fanciful idea since it is only possible to teleport from A to B the state representing one or more particles in A, by transferring it to one or more particles already existing in B.

One possible way of performing such a process consists in firstly learning all the properties of the original and unknown state, and then in transmitting them, by means of classical communication channels, to a receiver able to recreate a perfect copy.

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Nevertheless such a procedure would not work, for it would be necessary to have an infinite ensemble of identically prepared quantum systems in order to completely determine their original state.

However one of the most striking features of Quantum Mechanics, i.e. the entanglement, supplies us with a quantum channel of communication being able to transfer an unknown quantum state from a place to another, without knowing it and without violating special relativity constraints (i.e., not instantaneously).

Following the original proposal of Bennett et al. [1], to teleport a single-particle state the sender, traditionally named Alice, and the receiver, Bob, need only to share a particular two-particle maximally entangled state (an EPR singlet state) which acts as a purely quantum channel, and a telephone, the classical channel.

The core of the teleportation process resides on a projective local measurement, performed by Alice, in the Bell basis consisting of four orthonormal and maximally entangled two-particle states, involving the unknown one. After sending the result of the measurement by the classical channel (the crucial step which prevents from sending faster-than light messages) Bob can reconstruct, applying the unitary and local transformations suggested by Alice, a perfect copy of the original state. At the end the unknown state of Alice is destroyed, respecting in a such way the no-cloning theorem [2].

Bennett et al. applied this type of teleportation to single-particle states suggesting however that it should have been working also for arbitrary N -particle entangled states. As a matter of fact, it works through the use of N two-particle quantum channels.

In the present work we explore the possibility of obtaining the same result using only one quantum channel, realized by peculiar kinds of three-particle states, shared by Alice and Bob: the sender possesses one of the particles while the remaining two, belonging to the receiver, will be used as building blocks for the copy of the teleported state. We note that three-particle states have been obtained experimentally and could be used in the present context [3].

As it will appear, only some kinds of two-particle entangled states, and not the most general one, can be teleported by means of three-particle quantum channels and measurements onto two-particle Bell entangled states.

In appendix A all the different ways to achieve permitted teleportation are enumerated.

2 Teleportation of an arbitrary state

Let us suppose that Alice wants to teleport to Bob, spacelike separated from her, an unknown and arbitrary two-particle entangled state:

$$\begin{cases} |\psi\rangle_{12} = \alpha|00\rangle_{12} + \beta|10\rangle_{12} + \delta|01\rangle_{12} + \gamma|11\rangle_{12} \\ |\alpha|^2 + |\beta|^2 + |\delta|^2 + |\gamma|^2 = 1 \end{cases} \quad (2.1)$$

We are following the Quantum Information Theory convention of indicating with $\{|0\rangle, |1\rangle\}$ a complete orthonormal basis for a two dimensional, single particle, Hilbert space, made up of eigenstates of operator σ_z with eigenvalues respectively $+1$ and -1 . In order to teleport a two-particle state, the simplest single quantum-channel which can be used must be a three-particle one:

$$|\phi\rangle_{345} = \frac{1}{\sqrt{N}} [a|000\rangle + b|100\rangle + c|010\rangle + d|001\rangle + e|110\rangle + f|101\rangle + g|011\rangle + h|111\rangle]_{345} \quad (2.2)$$

with $a, b, \dots, h = \pm 1$ or 0 , and N equal to the number of coefficients which are not zeros.

The initial state we start from, will be the product of the two states just defined :

$$|\Omega\rangle = |\psi\rangle_{12} |\phi\rangle_{345} \quad (2.3)$$

where particles labelled 1, 2 and 3 belong to Alice (she can perform local measurements only on them), while particles labelled 4 and 5 are placed near Bob and they will constitute the bricks to make a copy of the state $|\psi\rangle$.

The core of the teleportation mechanism resides in performing all the direct products between states in $|\psi\rangle_{12}$ and $|\phi\rangle_{345}$ and in expressing subsequently the Alice's states of particles 2 and 3 in terms of vectors of the Bell basis.

This complete basis consists of four orthonormal maximally entangled states, simultaneous eigenvectors of the two commuting operators, $(\sigma_x)_2(\sigma_x)_3$ and $(\sigma_z)_2(\sigma_z)_3$:

$$\begin{aligned} |\phi^+\rangle_{23} &= \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)_{23} & |\psi^+\rangle_{23} &= \frac{1}{\sqrt{2}} (|01\rangle + |10\rangle)_{23} \\ |\phi^-\rangle_{23} &= \frac{1}{\sqrt{2}} (|00\rangle - |11\rangle)_{23} & |\psi^-\rangle_{23} &= \frac{1}{\sqrt{2}} (|01\rangle - |10\rangle)_{23} \end{aligned} \quad (2.4)$$

Inserting expressions (2.4) into equation (2.3) the resulting state of the system – completely equivalent to the original one since no physical process has taken place – is :

$$\begin{aligned} |\Omega\rangle &= \frac{1}{\sqrt{2N}} |\phi^+\rangle_{23} \{ |0\rangle_1 [(\alpha a + \delta b)|00\rangle + (\alpha c + \delta e)|10\rangle + (\alpha d + \delta f)|01\rangle + (\alpha g + \delta h)|11\rangle]_{45} + \\ &\quad + |1\rangle_1 [(\beta a + \gamma b)|00\rangle + (\beta c + \gamma e)|10\rangle + (\beta d + \gamma f)|01\rangle + (\beta g + \gamma h)|11\rangle]_{45} \} + \\ &\frac{1}{\sqrt{2N}} |\phi^-\rangle_{23} \{ |0\rangle_1 [(\alpha a - \delta b)|00\rangle + (\alpha c - \delta e)|10\rangle + (\alpha d - \delta f)|01\rangle + (\alpha g - \delta h)|11\rangle]_{45} + \\ &\quad + |1\rangle_1 [(\beta a - \gamma b)|00\rangle + (\beta c - \gamma e)|10\rangle + (\beta d - \gamma f)|01\rangle + (\beta g - \gamma h)|11\rangle]_{45} \} + \\ &\frac{1}{\sqrt{2N}} |\psi^+\rangle_{23} \{ |0\rangle_1 [(\alpha b + \delta a)|00\rangle + (\alpha e + \delta c)|10\rangle + (\alpha f + \delta d)|01\rangle + (\alpha h + \delta g)|11\rangle]_{45} + \\ &\quad + |1\rangle_1 [(\beta b + \gamma a)|00\rangle + (\beta e + \gamma c)|10\rangle + (\beta f + \gamma d)|01\rangle + (\beta h + \gamma g)|11\rangle]_{45} \} + \\ &\frac{1}{\sqrt{2N}} |\psi^-\rangle_{23} \{ |0\rangle_1 [(\alpha b - \delta a)|00\rangle + (\alpha e - \delta c)|10\rangle + (\alpha f - \delta d)|01\rangle + (\alpha h - \delta g)|11\rangle]_{45} + \\ &\quad + |1\rangle_1 [(\beta b - \gamma a)|00\rangle + (\beta e - \gamma c)|10\rangle + (\beta f - \gamma d)|01\rangle + (\beta h - \gamma g)|11\rangle]_{45} \} \end{aligned} \quad (2.5)$$

Alice's strategy to pursue the teleportation process, as already said, will consist in a local projective measurement onto the vectors of the Bell basis of particles 2 and 3 and in a successive measurement on particle 1, to be specified, in order to leave Bob's particles 4 and 5 in a state looking very much like Alice's original one.

As we are going to show, it turns out that there is no possible choice of the coefficients a, b, \dots, h , up to now deliberately unspecified, fulfilling the desired task of complete teleportation : it will not be possible for Bob to reconstruct the state $|\psi\rangle$.

First of all it is worthwhile observing that in equation (2.5) the couples of coefficients (α, δ) and (β, γ) are only and always associated to states $|0\rangle_1$ and $|1\rangle_1$, respectively.

It is so clear that successive measurement on Bell basis for particles 2 and 3 and on canonical basis of particle 1, leave the collapsed state quite dissimilar from the one to teleport, because of the lack of needed coefficients.

Suppose for example Alice obtains the state $|\phi^+\rangle_{23}$ and the state $|0\rangle_1$ in successive measurements: in the remaining state of the system there will be no trace of β and γ , preventing Bob from reconstructing state $|\psi\rangle$. In order to overcome this difficulty, Alice must resort to perform an Hadamard transformation on particle 1, so mixing up the coefficients.

We note that Alice could obtain the same result by making the substitution :

$$\begin{aligned} |0\rangle_1 &= A|\varphi\rangle_1 + B|\chi\rangle_1 \\ |1\rangle_1 &= -B^*|\varphi\rangle_1 + A^*|\chi\rangle_1 \end{aligned} \quad (2.6)$$

In fact, in order not to have undesired coefficients in the transmitted state, she should be obliged to choose $A=B=1/\sqrt{2}$, obtaining an expression having exactly the same distinctive features of equation (2.9). This means that making the Hadamard transformation or projecting on states $|\varphi_1\rangle_1$ and $|\chi_1\rangle_1$ has the same effect.

Hadamard unitary transformation is defined in terms of Pauli matrices as :

$$H = \frac{1}{\sqrt{2}} (\sigma_x + \sigma_z) \quad H^2 = 1 \quad (2.7)$$

It acts as a rotation about the axis $\vec{n} = 1/\sqrt{2} (\vec{n}_x + \vec{n}_z)$:

$$\begin{cases} H|0\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \\ H|1\rangle = \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle) \end{cases} \quad (2.8)$$

After applying such a transformation, the resulting state of the system with mixed coefficients, is :

$$\begin{aligned} |\tilde{\Omega}\rangle &= \frac{1}{2\sqrt{N}} |\phi^+\rangle_{23} \{ |0\rangle_1 [((\alpha + \beta)a + (\delta + \gamma)b) |00\rangle + ((\alpha + \beta)c + (\delta + \gamma)e) |10\rangle + \\ &\quad + ((\alpha + \beta)d + (\delta + \gamma)f) |01\rangle + ((\alpha + \beta)g + (\delta + \gamma)h) |11\rangle]_{45} + \\ &\quad |1\rangle_1 [((\alpha - \beta)a + (\delta - \gamma)b) |00\rangle + ((\alpha - \beta)c + (\delta - \gamma)e) |10\rangle + \\ &\quad + ((\alpha - \beta)d + (\delta - \gamma)f) |01\rangle + ((\alpha - \beta)g + (\delta - \gamma)h) |11\rangle]_{45} \} + \\ &\quad \frac{1}{2\sqrt{N}} |\phi^-\rangle_{23} \{ |0\rangle_1 [((\alpha + \beta)a - (\delta + \gamma)b) |00\rangle + ((\alpha + \beta)c - (\delta + \gamma)e) |10\rangle + \\ &\quad + ((\alpha + \beta)d - (\delta + \gamma)f) |01\rangle + ((\alpha + \beta)g - (\delta + \gamma)h) |11\rangle]_{45} + \\ &\quad |1\rangle_1 [((\alpha - \beta)a - (\delta - \gamma)b) |00\rangle + ((\alpha - \beta)c - (\delta - \gamma)e) |10\rangle + \\ &\quad + ((\alpha - \beta)d - (\delta - \gamma)f) |01\rangle + ((\alpha - \beta)g - (\delta - \gamma)h) |11\rangle]_{45} \} + \\ &\quad \frac{1}{2\sqrt{N}} |\psi^+\rangle_{23} \{ |0\rangle_1 [((\alpha + \beta)b + (\delta + \gamma)a) |00\rangle + ((\alpha + \beta)e + (\delta + \gamma)c) |10\rangle + \\ &\quad + ((\alpha + \beta)f + (\delta + \gamma)d) |01\rangle + ((\alpha + \beta)h + (\delta + \gamma)g) |11\rangle]_{45} + \\ &\quad |1\rangle_1 [((\alpha - \beta)b + (\delta - \gamma)a) |00\rangle + ((\alpha - \beta)e + (\delta - \gamma)c) |10\rangle + \end{aligned}$$

$$\begin{aligned}
& + ((\alpha - \beta)f + (\delta - \gamma)d) |01\rangle + ((\alpha - \beta)h + (\delta - \gamma)g) |11\rangle]_{45}\} + \\
& \frac{1}{2\sqrt{N}} |\psi^-\rangle_{23} \{ |0\rangle_1 [((\alpha + \beta)b - (\delta + \gamma)a) |00\rangle + ((\alpha + \beta)e - (\delta + \gamma)c) |10\rangle + \\
& + ((\alpha + \beta)f - (\delta + \gamma)d) |01\rangle + ((\alpha + \beta)h - (\delta + \gamma)g) |11\rangle]_{45} + \\
& |1\rangle_1 [((\alpha - \beta)b - (\delta - \gamma)a) |00\rangle + ((\alpha - \beta)e - (\delta - \gamma)c) |10\rangle + \\
& + ((\alpha - \beta)f - (\delta - \gamma)d) |01\rangle + ((\alpha - \beta)h - (\delta - \gamma)g) |11\rangle]_{45} \}
\end{aligned} \tag{2.9}$$

Alice now accomplishes Bell measurement on particles 2 and 3, and canonical measurement on particle 1, projecting in a such way the state firstly on Bell basis $\{|\phi^\pm\rangle, |\psi^\pm\rangle\}$ and secondly onto the canonical basis $\{|0\rangle, |1\rangle\}$: the state of the system will eventually collapse, with equal probability, in one of the eight states of particles 4 and 5 indicated between square brackets.

It is clear that there is no possible choice of a, b, \dots, h suited to reproduce the unknown initial state $|\psi\rangle$, because the coefficients $\alpha, \beta, \delta, \gamma$ – although mixed – remains summed two by two ($\alpha \pm \beta$ and $\delta \pm \gamma$).

Then, it is not possible to achieve complete teleportation of an arbitrary and unknown two-particle entangled state (i.e. with $\alpha, \beta, \gamma, \delta \neq 0$) using the present scheme.

One can then ask whether a measurement on orthonormal states $|\phi_i\rangle_{23}$, with $i = 1\dots 4$, different from the Bell states, could make one to achieve the desired goal. A cumbersome but trivial calculation shows that there is no gain on projecting onto general states and in all cases teleportation cannot be obtained.

In particular, since the proof is valid for every choice of non zero coefficients $\alpha, \beta, \gamma, \delta$, it is not even possible to teleport a two-particle factorized state using the present procedure. In fact, each state of the type $(m|0\rangle_1 + n|1\rangle_1)(r|0\rangle_2 + s|1\rangle_2)$ can be written in the form (2.1); conversely, it can be trivially shown that every state of the form (2.1), for which the condition $\alpha\delta = \gamma\beta$ is satisfied, can be expressed as a factorized state.

This result is not surprising. In fact, Alice has at her disposal only one particle, i.e. one e-bit, which is not sufficient to teleport a general state of two particles. However, in all these cases, one can obtain successful teleportation by simply repeating the original standard teleportation procedure using a sequence of two (two-particle) channels, rather than a single (three-particle) quantum channel as considered in this article.

3 Teleportation of peculiar states

Since the present mechanism for teleportation cannot work for an arbitrary two-particle entangled state, let us try to focus our attention on some two dimensional subspaces of the whole Hilbert space of the system of the two particles (which is four dimensional). Let us try for example with the state $\alpha|00\rangle + \gamma|11\rangle$.

Alice is now able to successfully perform the teleportation process by choosing a suitable quantum channel and then following the steps already considered in the previous section:

1. Alice prepares the state $|\Omega\rangle = (\alpha|00\rangle_{12} + \gamma|11\rangle_{12}) |\phi\rangle_{345}$ where $|\phi\rangle_{345}$ is the three-particle quantum channel, yet to be specified.

2. Alice acts with Hadamard unitary transformation on states of particle 1, in order to mix up in an appropriate way the coefficients α and γ .
3. Alice performs a Bell measurement on particles 2 and 3.
4. Alice performs a measurement onto states $|0\rangle_1$ and $|1\rangle_1$.

The eight equally probable results are easily obtained by putting $\beta=\delta=0$ in equation (2.9) :

- $|0\rangle_1|\phi^+\rangle_{23} \Rightarrow (\alpha a + \gamma b)|00\rangle_{45} + (\alpha c + \gamma e)|10\rangle_{45} + (\alpha d + \gamma f)|01\rangle_{45} + (\alpha g + \gamma h)|11\rangle_{45}$
- $|1\rangle_1|\phi^+\rangle_{23} \Rightarrow (\alpha a - \gamma b)|00\rangle_{45} + (\alpha c - \gamma e)|10\rangle_{45} + (\alpha d - \gamma f)|01\rangle_{45} + (\alpha g - \gamma h)|11\rangle_{45}$
- $|0\rangle_1|\phi^-\rangle_{23} \Rightarrow (\alpha a - \gamma b)|00\rangle_{45} + (\alpha c - \gamma e)|10\rangle_{45} + (\alpha d - \gamma f)|01\rangle_{45} + (\alpha g - \gamma h)|11\rangle_{45}$
- $|1\rangle_1|\phi^-\rangle_{23} \Rightarrow (\alpha a + \gamma b)|00\rangle_{45} + (\alpha c + \gamma e)|10\rangle_{45} + (\alpha d + \gamma f)|01\rangle_{45} + (\alpha g + \gamma h)|11\rangle_{45}$
- $|0\rangle_1|\psi^+\rangle_{23} \Rightarrow (\alpha b + \gamma a)|00\rangle_{45} + (\alpha e + \gamma c)|10\rangle_{45} + (\alpha f + \gamma d)|01\rangle_{45} + (\alpha h + \gamma g)|11\rangle_{45}$
- $|1\rangle_1|\psi^+\rangle_{23} \Rightarrow (\alpha b - \gamma a)|00\rangle_{45} + (\alpha e - \gamma c)|10\rangle_{45} + (\alpha f - \gamma d)|01\rangle_{45} + (\alpha h - \gamma g)|11\rangle_{45}$
- $|0\rangle_1|\psi^-\rangle_{23} \Rightarrow (\alpha b - \gamma a)|00\rangle_{45} + (\alpha e - \gamma c)|10\rangle_{45} + (\alpha f - \gamma d)|01\rangle_{45} + (\alpha h - \gamma g)|11\rangle_{45}$
- $|1\rangle_1|\psi^-\rangle_{23} \Rightarrow (\alpha b + \gamma a)|00\rangle_{45} + (\alpha e + \gamma c)|10\rangle_{45} + (\alpha f + \gamma d)|01\rangle_{45} + (\alpha h + \gamma g)|11\rangle_{45}$

There are now eight possible ways of choosing coefficients a, b, \dots, h – so there are eight quantum channels shared between Alice and Bob – and, correspondingly, there are eight different sets of instructions to send via classical communications to Bob, in order to complete the teleportation process (see Appendix A).

After having received that, Bob can successfully reconstruct the original state by applying local unitary transformations on his particles.

Let us focus for example on the choice $a = h = 1$, $b = c = d = e = f = g = 0$, corresponding to the quantum channel $|\phi\rangle = 1/\sqrt{2}(|000\rangle + |111\rangle)$. We list below the set of instructions for Bob according to the results of Alice's measurements :

Alice's measurements	Bob's states	Bob's instructions
$ 0\rangle_1 \phi^+\rangle_{23}$	$\alpha 00\rangle_{45} + \gamma 11\rangle_{45}$	do nothing
$ 1\rangle_1 \phi^+\rangle_{23}$	$\alpha 00\rangle_{45} - \gamma 11\rangle_{45}$	apply $(\sigma_z)_4 \otimes I_5$
$ 0\rangle_1 \phi^-\rangle_{23}$	$\alpha 00\rangle_{45} - \gamma 11\rangle_{45}$	apply $I_4 \otimes (\sigma_z)_5$
$ 1\rangle_1 \phi^-\rangle_{23}$	$\alpha 00\rangle_{45} + \gamma 11\rangle_{45}$	do nothing
$ 0\rangle_1 \psi^+\rangle_{23}$	$\gamma 00\rangle_{45} + \alpha 11\rangle_{45}$	apply $(\sigma_x)_4 \otimes (\sigma_x)_5$
$ 1\rangle_1 \psi^+\rangle_{23}$	$-\gamma 00\rangle_{45} + \alpha 11\rangle_{45}$	apply $(\sigma_z\sigma_x)_4 \otimes (\sigma_x)_5$
$ 0\rangle_1 \psi^-\rangle_{23}$	$-\gamma 00\rangle_{45} + \alpha 11\rangle_{45}$	apply $(\sigma_x)_4 \otimes (\sigma_z\sigma_x)_5$
$ 1\rangle_1 \psi^-\rangle_{23}$	$\gamma 00\rangle_{45} + \alpha 11\rangle_{45}$	apply $(\sigma_x)_4 \otimes (\sigma_x)_5$

The remaining seven possible quantum channels and seven sets of instructions are listed in Appendix A. As it is shown in the same appendix, there are eight possible ways also for teleporting the state $\beta|10\rangle + \delta|01\rangle$, but only four possible ways for teleporting factorizable (non entangled) states like $\beta|10\rangle + \gamma|11\rangle$ and $\alpha|00\rangle + \delta|01\rangle$.

It is also possible to show that the machinery doesn't work on right-factorizable states like $\alpha|00\rangle + \beta|10\rangle$ and $\delta|01\rangle + \gamma|11\rangle$ (see appendix B).

Before concluding this section it is worthwhile summarizing the obtained results in a table :

Two-particle states	Can be teleported?
$\alpha 00\rangle_{12} + \beta 10\rangle_{12} + \delta 01\rangle_{12} + \gamma 11\rangle_{12}$	No
$\alpha 00\rangle + \gamma 11\rangle$	Yes, in eight different ways
$\beta 10\rangle + \delta 01\rangle$	Yes, in eight different ways
$\beta 10\rangle + \gamma 11\rangle$	Yes, in four different ways
$\alpha 00\rangle + \delta 01\rangle$	Yes, in four different ways
$\alpha 00\rangle + \beta 10\rangle$	No
$\delta 01\rangle + \gamma 11\rangle$	No

4 Conclusions

In this work it has been shown that the most general and unknown two-particle entangled state (i.e. the state $|\psi\rangle = \alpha|00\rangle + \beta|10\rangle + \delta|01\rangle + \gamma|11\rangle$ with $\alpha, \beta, \gamma, \delta \neq 0$) cannot be teleported using only one (three-particle) channel and Bell measurements.

We have nevertheless shown that some two-particle entangled states, belonging to two dimensional subspaces of the whole Hilbert space, can be successfully teleported from Alice to Bob using suitable and different three-particle quantum channels, with the aim of Hadamard transformation, Bell measurements and classical communication.

We have listed which are the states and the sets of unitary transformations to be performed by Bob in order to recreate a perfect copy of the original state, without violating special relativity constraints (classical communication prevents in fact from sending faster-than-light messages) and no-cloning theorem (the original state possessed by Alice is destroyed by Bell measurement).

Appendix A

Let us ask which are the states transmitted from Alice to Bob permitting him to reconstruct the original one. The operations Bob may use are :

1. to do nothing, if the teleported state is already the original one;
2. to make an exchange $|0\rangle \leftrightarrow |1\rangle$ on particle 4 or on particle 5, or on both;
3. to make a transformation of the *CNOT* type;
4. to use a product of $|0\rangle \leftrightarrow |1\rangle$ exchange and *CNOT* transformation.

The unitary operator *CNOT*, which acts on two-particle states by reversing the second entry if the first is 1, is defined as :

$$\begin{aligned}
CNOT|00\rangle &= |00\rangle & CNOT|01\rangle &= |01\rangle \\
CNOT|10\rangle &= |11\rangle & CNOT|11\rangle &= |10\rangle
\end{aligned} \tag{4.1}$$

Then, we can find out the states sent to Bob permitting teleportation by simply applying all the possible inverse operations to the original state. The possibilities, together with the operations which must be done by Bob, are listed below.

$ 00\rangle_{45}$	$ 10\rangle_{45}$	$ 01\rangle_{45}$	$ 11\rangle_{45}$	Bob's instructions
α	β	δ	γ	do nothing
β	α	γ	δ	apply $(\sigma_x)_4$
δ	γ	α	β	apply $(\sigma_x)_5$
γ	δ	β	α	apply $(\sigma_x)_4 \otimes (\sigma_x)_5$
α	γ	δ	β	apply $CNOT$
β	δ	γ	α	$(\sigma_x)_4 CNOT$
δ	β	α	γ	$(\sigma_x)_5 CNOT$
γ	α	β	δ	apply $(\sigma_x)_4 \otimes (\sigma_x)_5 CNOT$

It turns out that there are eight possible channels for transmitting states $\alpha|00\rangle + \gamma|11\rangle$ and $\beta|10\rangle + \delta|01\rangle$. In fact, putting $(\beta = 0, \delta = 0)$ or $(\alpha = 0, \gamma = 0)$ respectively, we obtain eight different transmitted states. In the case of states $\alpha|00\rangle + \delta|01\rangle$ and $\beta|10\rangle + \gamma|11\rangle$, the annihilation of coefficients β, γ and α, δ reduces to four the different states from which Bob can restore the original ones.

We are now going to list other seven possible quantum channels $|\phi\rangle_{345}$ and relative sets of instructions, with whom Alice and Bob can accomplish successful teleportation of the particular state $\alpha|00\rangle + \gamma|11\rangle$.

The various channels, being characterized by different choices of the coefficients a, b, \dots, h to be inserted in equation (2.9), are indicated in the following schemes together with the results of Alice's measurements, the collapsed state of particles 4 and 5 and the unitary transformation which Bob must perform in order to complete teleportation process.

Quantum Channels

1. Quantum Channel $|010\rangle_{345} + |101\rangle_{345}$ $c = f = 1, \quad a = b = d = e = g = h = 0$

Alice's measurements	Bob's states	Bob's instructions
$ 0\rangle_1 \phi^+\rangle_{23}$	$\alpha 10\rangle_{45} + \gamma 01\rangle_{45}$	apply $(\sigma_x)_4 \otimes I_5$
$ 1\rangle_1 \phi^+\rangle_{23}$	$\alpha 10\rangle_{45} - \gamma 01\rangle_{45}$	apply $(\sigma_x)_4 \otimes (\sigma_z)_5$
$ 0\rangle_1 \phi^-\rangle_{23}$	$\alpha 10\rangle_{45} - \gamma 01\rangle_{45}$	apply $(\sigma_x)_4 \otimes (\sigma_z)_5$
$ 1\rangle_1 \phi^-\rangle_{23}$	$\alpha 10\rangle_{45} + \gamma 01\rangle_{45}$	apply $(\sigma_x)_4 \otimes I_5$
$ 0\rangle_1 \psi^+\rangle_{23}$	$\gamma 10\rangle_{45} + \alpha 01\rangle_{45}$	apply $I_4 \otimes (\sigma_x)_5$
$ 1\rangle_1 \psi^+\rangle_{23}$	$-\gamma 10\rangle_{45} + \alpha 01\rangle_{45}$	apply $(\sigma_z)_4 \otimes (\sigma_x)_5$
$ 0\rangle_1 \psi^-\rangle_{23}$	$-\gamma 10\rangle_{45} + \alpha 01\rangle_{45}$	apply $(\sigma_z)_4 \otimes (\sigma_x)_5$
$ 1\rangle_1 \psi^-\rangle_{23}$	$\gamma 10\rangle_{45} + \alpha 01\rangle_{45}$	apply $I_4 \otimes (\sigma_x)_5$

2. Quantum Channel $|100\rangle_{345} + |011\rangle_{345}$ $b = g = 1, a = c = d = e = f = h = 0$

Alice's measurements	Bob's states	Bob's instructions
$ 0\rangle_1 \phi^+\rangle_{23}$	$\gamma 00\rangle_{45} + \alpha 11\rangle_{45}$	apply $(\sigma_x)_4 \otimes (\sigma_x)_5$
$ 1\rangle_1 \phi^+\rangle_{23}$	$-\gamma 00\rangle_{45} + \alpha 11\rangle_{45}$	apply $(\sigma_z \sigma_x)_4 \otimes (\sigma_x)_5$
$ 0\rangle_1 \phi^-\rangle_{23}$	$-\gamma 00\rangle_{45} + \alpha 11\rangle_{45}$	apply $(\sigma_z \sigma_x)_4 \otimes (\sigma_x)_5$
$ 1\rangle_1 \phi^-\rangle_{23}$	$\gamma 00\rangle_{45} + \alpha 11\rangle_{45}$	apply $(\sigma_x)_4 \otimes (\sigma_x)_5$
$ 0\rangle_1 \psi^+\rangle_{23}$	$\alpha 00\rangle_{45} + \gamma 11\rangle_{45}$	do nothing
$ 1\rangle_1 \psi^+\rangle_{23}$	$\alpha 00\rangle_{45} - \gamma 11\rangle_{45}$	apply $I_4 \otimes (\sigma_z)_5$
$ 0\rangle_1 \psi^-\rangle_{23}$	$\alpha 00\rangle_{45} - \gamma 11\rangle_{45}$	apply $I_4 \otimes (\sigma_z)_5$
$ 1\rangle_1 \psi^-\rangle_{23}$	$\alpha 00\rangle_{45} + \gamma 11\rangle_{45}$	do nothing

3. Quantum Channel $|110\rangle_{345} + |001\rangle_{345}$ $d = e = 1, a = b = c = f = g = h = 0$

Alice's measurements	Bob's states	Bob's instructions
$ 0\rangle_1 \phi^+\rangle_{23}$	$\gamma 10\rangle_{45} + \alpha 01\rangle_{45}$	apply $I_4 \otimes (\sigma_x)_5$
$ 1\rangle_1 \phi^+\rangle_{23}$	$-\gamma 10\rangle_{45} + \alpha 01\rangle_{45}$	apply $(\sigma_z)_4 \otimes (\sigma_x)_5$
$ 0\rangle_1 \phi^-\rangle_{23}$	$-\gamma 10\rangle_{45} + \alpha 01\rangle_{45}$	apply $(\sigma_z)_4 \otimes (\sigma_x)_5$
$ 1\rangle_1 \phi^-\rangle_{23}$	$\gamma 10\rangle_{45} + \alpha 01\rangle_{45}$	apply $I_4 \otimes (\sigma_x)_5$
$ 0\rangle_1 \psi^+\rangle_{23}$	$\alpha 10\rangle_{45} + \gamma 01\rangle_{45}$	apply $(\sigma_x)_4 \otimes I_5$
$ 1\rangle_1 \psi^+\rangle_{23}$	$\alpha 10\rangle_{45} - \gamma 01\rangle_{45}$	apply $(\sigma_x)_4 \otimes (\sigma_z)_5$
$ 0\rangle_1 \psi^-\rangle_{23}$	$\alpha 10\rangle_{45} - \gamma 01\rangle_{45}$	apply $(\sigma_x)_4 \otimes (\sigma_z)_5$
$ 1\rangle_1 \psi^-\rangle_{23}$	$\alpha 10\rangle_{45} + \gamma 01\rangle_{45}$	apply $(\sigma_x)_4 \otimes I_5$

4. Quantum Channel $|000\rangle_{345} + |110\rangle_{345}$ $a = e = 1, b = c = d = f = g = h = 0$

Alice's measurements	Bob's states	Bob's instructions
$ 0\rangle_1 \phi^+\rangle_{23}$	$\alpha 00\rangle_{45} + \gamma 10\rangle_{45}$	apply $CNOT$
$ 1\rangle_1 \phi^+\rangle_{23}$	$\alpha 00\rangle_{45} - \gamma 10\rangle_{45}$	apply $(\sigma_z)_5 CNOT$
$ 0\rangle_1 \phi^-\rangle_{23}$	$\alpha 00\rangle_{45} - \gamma 10\rangle_{45}$	apply $(\sigma_z)_5 CNOT$
$ 0\rangle_1 \phi^-\rangle_{23}$	$\alpha 00\rangle_{45} + \gamma 10\rangle_{45}$	apply $CNOT$
$ 0\rangle_1 \psi^+\rangle_{23}$	$\gamma 00\rangle_{45} + \alpha 10\rangle_{45}$	apply $(\sigma_x)_4 \otimes (\sigma_x)_5 CNOT$
$ 1\rangle_1 \psi^+\rangle_{23}$	$-\gamma 00\rangle_{45} + \alpha 10\rangle_{45}$	apply $(\sigma_z \sigma_x)_4 \otimes (\sigma_x)_5 CNOT$
$ 0\rangle_1 \psi^-\rangle_{23}$	$-\gamma 00\rangle_{45} + \alpha 10\rangle_{45}$	apply $(\sigma_z \sigma_x)_4 \otimes (\sigma_x)_5 CNOT$
$ 1\rangle_1 \psi^-\rangle_{23}$	$\gamma 00\rangle_{45} + \alpha 10\rangle_{45}$	apply $(\sigma_x)_4 \otimes (\sigma_x)_5 CNOT$

5. Quantum Channel $|100\rangle_{345} + |010\rangle_{345}$ $b = c = 1, a = d = e = f = g = h = 0$

Alice's measurements	Bob's states	Bob's instructions
$ 0\rangle_1 \phi^+\rangle_{23}$	$\gamma 00\rangle_{45} + \alpha 10\rangle_{45}$	apply $(\sigma_x)_4 \otimes (\sigma_x)_5 CNOT$
$ 1\rangle_1 \phi^+\rangle_{23}$	$-\gamma 00\rangle_{45} + \alpha 10\rangle_{45}$	apply $(\sigma_z \sigma_x)_4 \otimes (\sigma_x)_5 CNOT$
$ 0\rangle_1 \phi^-\rangle_{23}$	$-\gamma 00\rangle_{45} + \alpha 10\rangle_{45}$	apply $(\sigma_z \sigma_x)_4 \otimes (\sigma_x)_5 CNOT$
$ 1\rangle_1 \phi^-\rangle_{23}$	$\gamma 00\rangle_{45} + \alpha 10\rangle_{45}$	apply $(\sigma_x)_4 \otimes (\sigma_x)_5 CNOT$
$ 0\rangle_1 \psi^+\rangle_{23}$	$\alpha 00\rangle_{45} + \gamma 10\rangle_{45}$	apply $CNOT$
$ 1\rangle_1 \psi^+\rangle_{23}$	$\alpha 00\rangle_{45} - \gamma 10\rangle_{45}$	apply $(\sigma_z)_5 CNOT$
$ 0\rangle_1 \psi^-\rangle_{23}$	$\alpha 00\rangle_{45} - \gamma 10\rangle_{45}$	apply $(\sigma_z)_5 CNOT$
$ 0\rangle_1 \psi^-\rangle_{23}$	$\alpha 00\rangle_{45} + \gamma 10\rangle_{45}$	apply $CNOT$

6. Quantum Channel $|101\rangle_{345} + |011\rangle_{345}$ $f = g = 1, a = b = c = d = e = h = 0$

Alice's measurements	Bob's states	Bob's instructions
$ 0\rangle_1 \phi^+\rangle_{23}$	$\gamma 01\rangle_{45} + \alpha 11\rangle_{45}$	apply $(\sigma_x)_4 CNOT$
$ 1\rangle_1 \phi^+\rangle_{23}$	$-\gamma 01\rangle_{45} + \alpha 11\rangle_{45}$	apply $(\sigma_z)_5 \otimes (\sigma_x)_4 CNOT$
$ 0\rangle_1 \phi^-\rangle_{23}$	$-\gamma 01\rangle_{45} + \alpha 11\rangle_{45}$	apply $(\sigma_z)_5 \otimes (\sigma_x)_4 CNOT$
$ 1\rangle_1 \phi^-\rangle_{23}$	$\gamma 01\rangle_{45} + \alpha 11\rangle_{45}$	apply $(\sigma_x)_4 CNOT$
$ 0\rangle_1 \psi^+\rangle_{23}$	$\alpha 01\rangle_{45} + \gamma 11\rangle_{45}$	apply $(\sigma_x)_5 CNOT$
$ 1\rangle_1 \psi^+\rangle_{23}$	$\alpha 01\rangle_{45} - \gamma 11\rangle_{45}$	apply $(\sigma_z)_4 \otimes (\sigma_x)_5 CNOT$
$ 0\rangle_1 \psi^-\rangle_{23}$	$\alpha 01\rangle_{45} - \gamma 11\rangle_{45}$	apply $(\sigma_z)_4 \otimes (\sigma_x)_5 CNOT$
$ 0\rangle_1 \psi^-\rangle_{23}$	$\alpha 01\rangle_{45} + \gamma 11\rangle_{45}$	apply $(\sigma_x)_5 CNOT$

7. Quantum Channel $|001\rangle_{345} + |111\rangle_{345}$ $d = h = 1, a = b = c = e = f = g = 0$

Alice's measurements	Bob's states	Bob's instructions
$ 0\rangle_1 \phi^+\rangle_{23}$	$\alpha 01\rangle_{45} + \gamma 11\rangle_{45}$	apply $(\sigma_x)_5 CNOT$
$ 1\rangle_1 \phi^+\rangle_{23}$	$\alpha 01\rangle_{45} - \gamma 11\rangle_{45}$	apply $(\sigma_z)_4 \otimes (\sigma_x)_5 CNOT$
$ 0\rangle_1 \phi^-\rangle_{23}$	$\alpha 01\rangle_{45} - \gamma 11\rangle_{45}$	apply $(\sigma_z)_4 \otimes (\sigma_x)_5 CNOT$
$ 0\rangle_1 \phi^-\rangle_{23}$	$\alpha 01\rangle_{45} + \gamma 11\rangle_{45}$	apply $(\sigma_x)_5 CNOT$
$ 0\rangle_1 \psi^+\rangle_{23}$	$\gamma 01\rangle_{45} + \alpha 11\rangle_{45}$	apply $(\sigma_x)_4 CNOT$
$ 1\rangle_1 \psi^+\rangle_{23}$	$-\gamma 01\rangle_{45} + \alpha 11\rangle_{45}$	apply $(\sigma_z)_5 \otimes (\sigma_x)_4 CNOT$
$ 0\rangle_1 \psi^-\rangle_{23}$	$-\gamma 01\rangle_{45} + \alpha 11\rangle_{45}$	apply $(\sigma_z)_5 \otimes (\sigma_x)_4 CNOT$
$ 1\rangle_1 \psi^-\rangle_{23}$	$\gamma 01\rangle_{45} + \alpha 11\rangle_{45}$	apply $(\sigma_x)_4 CNOT$

Appendix B

For the sake of simplicity we will only enumerate the permitted three-particle quantum channels that Alice and Bob may use to teleport the following two-particle states:

Two-particle states	Quantum channels
$[\beta 10\rangle + \delta 01\rangle]_{12}$	$(010\rangle + 101\rangle)_{345}$
	$(000\rangle + 111\rangle)_{345}$
	$(100\rangle + 011\rangle)_{345}$
	$(110\rangle + 001\rangle)_{345}$
	$(001\rangle + 111\rangle)_{345}$
	$(101\rangle + 011\rangle)_{345}$
	$(000\rangle + 110\rangle)_{345}$
	$(100\rangle + 010\rangle)_{345}$
$[\alpha 00\rangle + \delta 01\rangle]_{12}$	$(000\rangle + 110\rangle)_{345}$
	$(100\rangle + 010\rangle)_{345}$
	$(001\rangle + 111\rangle)_{345}$
	$(101\rangle + 011\rangle)_{345}$

Two-particle states	Quantum channels
$[\beta 10\rangle + \gamma 11\rangle]_{12}$	$(000\rangle + 110\rangle)_{345}$ $(100\rangle + 010\rangle)_{345}$ $(001\rangle + 111\rangle)_{345}$ $(101\rangle + 011\rangle)_{345}$

Two-particle factorized states in which the unknown part is in channel 1 (i.e., states $\alpha|00\rangle_{12} + \beta|10\rangle_{12} \equiv (\alpha|0\rangle_1 + \beta|1\rangle_1)|0\rangle_2$ and $\delta|01\rangle_{12} + \gamma|11\rangle_{12} \equiv (\delta|0\rangle_1 + \gamma|1\rangle_1)|1\rangle_2$ cannot be transmitted using the present method.

Let us consider for example the state $\alpha|00\rangle_{12} + \beta|10\rangle_{12}$; this means to choose $\gamma = 0$ and $\delta = 0$ in equation (2.9), which becomes :

$$|\tilde{\Omega}\rangle = \frac{1}{2\sqrt{N}} [(\alpha + \beta)|0\rangle_1 + (\alpha - \beta)|1\rangle_1] \cdot [|\phi^+\rangle_{23} + |\phi^-\rangle_{23} + |\psi^+\rangle_{23} + |\psi^-\rangle_{23}] \cdot [a|00\rangle_{45} + c|10\rangle_{45} + d|01\rangle_{45} + g|11\rangle_{45}] \quad (4.2)$$

and the teleporting method cannot be applied. The unknown state must be put in channel 2, which is involved in the Bell measurement. However, as already noted, in such a case one can resort to the original standard teleportation procedure.

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